

SOME RESULTS ON THE TOPOLOGY OF REAL BOTT TOWERS

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ABSTRACT. The main aim of this article is to study the topology of real Bott towers as special and interesting examples of real toric varieties. We first give a presentation of the fundamental group of a real Bott tower and show that the fundamental group is abelian if and only if the real Bott tower is a product of circles. We further prove that the fundamental group of a real Bott tower is always solvable and it is nilpotent if and only if it is abelian. We then describe the cohomology ring of a real Bott tower and also give recursive formulae for the Steifel Whitney classes. We derive combinatorial characterization for orientability of these manifolds and further give a combinatorial formula for the $(n - 1)$ th Steifel Whitney class. In particular, we show that if a Bott tower is orientable then the $(n - 1)$ th Steifel Whitney class must also vanish. Moreover, by deriving a combinatorial formula for the second Steifel-Whitney class we give a necessary and sufficient condition for the Bott tower to admit a spin structure. We finally prove the vanishing of all the Steifel-Whitney numbers and hence establish that these manifolds are null-cobordant.

1. INTRODUCTION

Bott towers are iterated fibre bundles with fibre at each stage being $\mathbb{P}_{\mathbb{C}}^1$. In particular they are smooth projective complex toric varieties. They were constructed in [13] by M.Grossberg and Y.Karshon who show that a Bott-Samelson variety can be deformed to a Bott tower. Bott-Samelson manifolds were first constructed in [3] to study cohomology of generalized flag varieties. M.Demazure and D.Hansen used it to obtain desingularizations of Schubert varieties in generalized flag varieties. Moreover, the underlying differentiable structure is preserved under the deformation. Because of their relation with Bott Samelson manifolds which in turn are related to the Schubert varieties, along with their amenable structure as iterated $\mathbb{P}_{\mathbb{C}}^1$ -bundles, the Bott towers have been important and interesting objects of study.

Topological invariants for these manifolds have been studied using their iterated sphere bundle structure. The equivariant cohomology of Bott Samelson manifolds have been studied with applications to the cohomology of Schubert varieties in [7]. Also see [24] for equivariant K -theory of Bott towers with application to the equivariant K -theory of flag manifolds.

Indeed, from the viewpoint of toric topology, Bott-towers can be seen to also have the structure of a quasi-toric manifold [6] with the quotient polytope being the n -dimensional cube I^n where n is the complex dimension of the Bott tower. The second named author and P.Sankaran described the topological K -ring of the quasi toric manifolds in [21], where the topological K -ring of Bott towers and Bott

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Samelson manifolds have been described as a special example. Also see [5], [4] and [1] for results on complex K -theory, KO -theory as well as the complex cobordism ring of Bott towers and quasi-toric manifolds.

Recently there has been extensive work on the topology and geometry of Bott towers viewed as a quasi-toric manifold (see for example [16], [10]). These works are especially related to the problem of cohomological rigidity of Bott manifolds or more generally of quasi-toric manifolds.

There has also been a parallel study on the topology of real Bott towers. These manifolds are constructed as iterated $\mathbb{P}_{\mathbb{R}}^1 = \mathbb{S}^1$ -bundles and can be viewed as a special example of a small cover defined by Davis and Januszkiewicz (see for example [15], [20], [9]).

The cohomology of a small cover have been described in [6]. Indeed in [6, Theorem 5.12] Davis and Januszkiewicz give a presentation for the cohomology ring with \mathbb{Z}_2 -coefficients as a quotient of the Stanley-Reisner ring of the simple convex polytope by certain canonical linear relations. A similar description for the cohomology ring with \mathbb{Z}_2 -coefficients of the real part of a smooth projective complex toric variety is due to Jurkiewicz (see [18, Theorem 4.3.1]).

In [6, Section 6], there is also a description of the tangent bundle of a small cover and a formula for its Steifel-Whitney class in terms of the generators of the cohomology ring. These generators are in turn the first Steifel-Whitney classes of certain canonical line bundles on the small cover.

In [22], the second named author of the current article described the fundamental group of a real toric variety associated to any smooth fan Δ in \mathbb{Z}^n . She in particular gave a presentation of the fundamental group, a combinatorial criterion on Δ for the fundamental group to be abelian along with a criterion for the real toric variety to be aspherical. The fundamental group of a small cover has also earlier been described in [6, Corollary 4.5] as the kernel of a natural map from the right-angled Coxeter group associated to the simple convex polytope P to \mathbb{Z}_2^n . The natural map is the composition of the characteristic map λ of the small cover with the abelianization map of the Coxeter group.

The authors in the current article were interested in the study of real Bott towers as a nice class of small covers or real toric varieties, which are characterized by the upper triangular real *Bott matrix* $C = (c_{i,j})$ with entries in \mathbb{Z}_2 (see definition below).

Our main motivation here is to give a precise and elegant description of all the above mentioned topological invariants namely, the fundamental group, cohomology ring and the Stiefel Whitney classes, in terms of the *Bott numbers* $\{c_{i,j}\}$. We exploit the inductive definition of this class of manifolds and derive precise topological information about them wherever possible. We further classify Bott towers satisfying a specific topological property like orientability or admission of spin structure, by means of certain algebraic identities on the $c_{i,j}$'s.

We now develop some notations before outlining our main results in the next section.

1.1. Notations and Conventions. In this section we recall the definition of a Bott tower and fix some notations (see [13]).

A *Bott tower* is a smooth complete complex toric variety which is constructed iteratively as follows:

Let $Y_1 = \mathbb{CP}^1$. Let L_2 be a holomorphic complex line bundle on \mathbb{CP}^1 . We then let $Y_2 = \mathbb{P}(\mathbf{1} \oplus L_2)$ where $\mathbf{1}$ is the trivial line bundle on \mathbb{CP}^1 . Then Y_2 is a \mathbb{CP}^1

bundle over \mathbb{CP}^1 which is a Hirzebruch surface. We can iterate this process for $2 \leq j \leq n$, where at each step, L_j is a complex line bundle over Y_{j-1} , and the variety $Y_j = \mathbb{P}(1 \oplus L_j)$ is a \mathbb{CP}^1 bundle over Y_{j-1} . The variety Y_n thus obtained after n -steps is called an n -step Bott tower.

Definition 1.1. *In fact an n -step Bott tower is a smooth complete toric variety of dimension n whose fan Δ can be described as follows:*

We take a collection of integers $\{c_{i,j}\}$, $1 \leq i < j \leq n$. Let e_1, e_2, \dots, e_n be the standard basis vectors of \mathbb{R}^n . Let $v_j = e_j$ for $1 \leq j \leq n$,

$$v_{n+j} = -e_j + \sum_{k=j+1}^n c_{j,k} \cdot e_k$$

for $1 \leq j \leq n-1$ and $v_{2n} = -e_n$. We define the fan Δ in \mathbb{R}^n consisting of cones generated by the set of vectors in any subcollection of $\{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{2n}\}$ which does not contain both v_i and v_{n+i} for $1 \leq i \leq n$.

Definition 1.2. *We can also view a Bott tower as a quasi-toric manifold (see [6]) over the n -cube I^n which is a simple convex polytope of dimension n . If we index the $2n$ facets of I^n by $F_1, F_2, \dots, F_n, F_{n+1}, \dots, F_{2n}$, then the characteristic function is defined on the collection of facets to \mathbb{Z}^n as follows: $\lambda(F_j) = e_j$ for $1 \leq j \leq n$,*

$$\lambda(F_{n+j}) = -e_j + \sum_{k=j+1}^n c_{j,k} \cdot e_{j+k}$$

for $1 \leq j \leq n-1$ and $\lambda(F_{2n}) = -e_n$.

1.1.1. Real Bott tower. We shall call the real part of the n -step complex Bott tower as the real n -step Bott tower.

In particular, $(Y_2)_{\mathbb{R}}$ is an \mathbb{RP}^1 bundle over $(Y_1)_{\mathbb{R}} = \mathbb{RP}^1$. Iteratively we construct $(Y_j)_{\mathbb{R}}$ as an \mathbb{RP}^1 bundle over $(Y_{j-1})_{\mathbb{R}}$ for $2 \leq j \leq n$. The real n -step Bott tower $(Y_n)_{\mathbb{R}}$ is indeed the real toric variety associated to the fan Δ described above (see [18, Section 2.4] and [22]).

Definition 1.3. *As in the complex case we can also view $(Y_n)_{\mathbb{R}}$ as a small cover over the simple convex polytope I^n , where the characteristic map λ is defined on the collection of facets \mathcal{F} to \mathbb{Z}_2^n as follows: $\lambda(F_j) = e_j$ for $1 \leq j \leq n$,*

$$\lambda(F_{n+j}) = e_j + \sum_{k=j+1}^n c_{j,k} \cdot e_k$$

for $1 \leq j \leq n-1$ and $\lambda(F_{2n}) = e_n$. Here $c_{i,j} \in \mathbb{Z}_2$ for $1 \leq i < j \leq n$. Thus $(Y_n)_{\mathbb{R}}$ is homeomorphic to the identification space $\mathbb{Z}_2^n \times I^n / \sim$ where $(t, p) \sim (t', p')$ if and only if $p = p'$ and $t \cdot (t')^{-1} \in G_{F(p)}$. Here $F(p) = F_1 \cap \dots \cap F_l$ is the unique face of I which contains p in its relative interior and $G_{F(p)}$ is the rank- l subgroup of \mathbb{Z}_2^n determined by the span of $\lambda(F_1), \dots, \lambda(F_l)$. Now, let $\pi : (Y_n)_{\mathbb{R}} \rightarrow I^n$ denote the second projection which maps $[t, p] \mapsto p$, and let $M_i := \pi^{-1}(F_i)$ denote the characteristic submanifold for $1 \leq i \leq 2n$. (See Section 1 of [6]).

The topological structure of an n -step real Bott tower is completely determined by the simple convex polytope I^n and the data encoded by the matrix

$$(1.1) \quad C = (c_{i,j}) \in M_n(\mathbb{Z}_2)$$

where $c_{i,i} = 1$ and $c_{i,j} = 0$ for $i > j$. Note that the i th row of C is $\lambda(F_{n+i}) \in \mathbb{Z}_2^n$ for $1 \leq i \leq n$. We call C the *Bott matrix*. Thus $Y_n = Y(C)$ the real Bott tower associated to C .

The 2-step real Bott tower is the torus or the Klein bottle depending on whether $c_{1,2} = 0$ or $c_{1,2} = 1$. The 3-step real Bott tower is an \mathbb{RP}^1 bundle over the torus or the Klein bottle whose topological structure depends on $c_{1,2}, c_{1,3}$ and $c_{2,3}$.

Notation 1.4. In this article, since we are mainly interested in the study of the real Bott tower, for notational simplicity we shall henceforth denote $(Y_n)_{\mathbb{R}}$ by Y_n .

1.2. Overview of the main results. In Section 2 we give a description of the fundamental group of the real Bott tower. In particular, we give a presentation of the fundamental group in terms of generators and relations in Theorem 2.1 and in Corollary 2.2 we prove that Y_n has abelian fundamental group if and only if Y_n is a product of circles. Further, in Proposition 2.6 we show that the commutator subgroup of $\pi_1(Y_n)$ is always abelian, so that $\pi_1(Y_n)$ is solvable. In Proposition 2.9 we prove that the fundamental group is nilpotent if and only if it is abelian. By abelianizing the fundamental group we further determine the first homology with integer coefficients $H_1(Y_n; \mathbb{Z})$ explicitly. We also conclude by induction that Y_n is always aspherical. The main tool used in this section is the presentation of the fundamental group of the real part of any smooth toric variety in [22], where combinatorial characterizations are also given for the fundamental group to be abelian and the manifold to be aspherical. However, as mentioned earlier, in the case of the real Bott tower the fundamental group gets a neater and simpler presentation in terms of the entries of the characterizing Bott matrix. Moreover, the presentation also enables us to make further conclusions about the group theoretic properties of the fundamental group of these special class of real toric varieties. Further, proofs of all results in this section except that of Theorem 2.1 which uses [22, Lemma 3.2], are made self-contained and specific to the case of the real Bott tower.

In Section 3 we study the cohomology ring of these manifolds with \mathbb{Z}_2 -coefficients. The tool here is to apply the description of the cohomology ring of small covers in [6] or else that of a real toric variety in [18, Section 2.4] to these class of manifolds as a special case. We describe the cohomology ring in terms of generators and relations in Theorem 3.17, where again the presentation given in terms of the entries of the Bott matrix becomes simple in this case due to the iterative structure of these manifolds.

In Section 4, using the presentation of the cohomology ring and applying the more general results of [6] we give explicit description of the Steifel-Whitney classes of Y_n which are smooth connected compact manifolds. The description is again in terms of the entries of the Bott matrix. More explicitly, in Theorem 4.1 we give an inductive formula for total Steifel-Whitney class as well as the k th Steifel-Whitney class of Y_n in terms of those of Y_{n-1} .

These results are applied to give a combinatorial characterisation for orientability of Y_n in Theorem 4.5, criterion for Y_n to admit a spin structure in Theorem 4.10 and a nice formula for $w_{n-1}(Y_n)$ in Corollary 4.3. We further show in Corollary 4.6 that if Y_n is orientable then $w_{n-1}(Y_n)$ also vanishes. We finally prove the vanishing of all Steifel-Whitney numbers of Y_n , which enables us to conclude in Theorem 4.24 that the real Bott towers are null cobordant.

We aim to continue these methods to prove results, for instance, about the immersion and embedding dimensions and parallelizability of these manifolds in future work.

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2. THE FUNDAMENTAL GROUP

In this section we shall get a presentation for the fundamental group of the real Bott tower $Y(\Delta) = Y_n$ where Δ is as in Definition 1.1.

For obtaining a presentation for $\pi_1(Y_n)$ we apply [22, Proposition 3.1]. We shall first fix some notations which are slightly modified from [22] suitable to our setting. Recall that we have an exact sequence

$$(2.2) \quad 0 \rightarrow \pi_1(Y_n) \rightarrow W(\Delta) \rightarrow \mathbb{Z}_2^n \rightarrow 0$$

where $W(\Delta)$ is the right angled Coxeter group associated to Δ with the following presentation:

$$W(\Delta) = \langle s_j \mid s_j^2, 1 \leq j \leq 2n \text{ and } (s_i s_j)^2 \text{ for all } 1 \leq i < j \leq 2n \text{ with } j \neq i + n \rangle.$$

The last arrow in the above exact sequence is obtained by composing the natural abelianization map from $W(\Delta)$ to \mathbb{Z}^{2n} with the characteristic map λ from \mathbb{Z}^{2n} to \mathbb{Z}^n .

Let $\alpha_j := s_j s_{j-n} s_{j-n+1}^{c_{j-n,j-n+1}} \cdots s_{j-n+k}^{c_{j-n,j-n+k}} \cdots s_n^{c_{j-n,n}}$ for all $n+1 \leq j \leq 2n-1$ and $1 \leq k \leq 2n-j$ and $\alpha_{2n} := s_{2n} s_n$.

Let $b_i^j := c_{j-n,i}$ for $j-n+1 \leq i \leq n$, $b_{j-n}^j = 1$ and $b_k^j = 0$ for $k < j-n$, for every $n+1 \leq j \leq 2n$. Thus

$$(2.3) \quad b^j = (b_i^j)_{i=1,\dots,n}$$

denotes the $(j-n)$ th row vector of the Bott matrix C .

For $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \mathbb{Z}_2^n$ where $\epsilon_i \in \{0, 1\}$, let

$$t_\epsilon := s_1^{\epsilon_1} \cdots s_n^{\epsilon_n}$$

in $W(\Delta)$.

Now, from the relations in $W(\Delta)$ it follows that:

$$(2.4) \quad t_\epsilon \alpha_j t_\epsilon = \begin{cases} \alpha_j & \text{if } \epsilon_{j-n} = 0 \\ \alpha_j^{-1} & \text{if } \epsilon_{j-n} = 1 \end{cases}$$

Since $b_{j-n}^j = 1$ and $b_{k-n}^j = 0$ for $k-n < j-n$ it follows that:

$$(2.5) \quad t_{bj} \alpha_j t_{bj} = \alpha_j^{-1}$$

for all $n+1 \leq j \leq 2n$ and

$$(2.6) \quad t_{bj} \alpha_k t_{bj} = \alpha_k$$

for $n+1 \leq k < j \leq 2n$. Moreover, since $b_{k-n}^j = c_{j-n,k-n}$ we have:

$$(2.7) \quad t_{bj} \alpha_k t_{bj} = \begin{cases} \alpha_k & \text{if } c_{j-n,k-n} = 0 \\ \alpha_k^{-1} & \text{if } c_{j-n,k-n} = 1 \end{cases}$$

for all $n+1 \leq j < k \leq 2n$.

Theorem 2.1. *We have a presentation for $\pi_1(Y_n) = \langle S \mid R \rangle$ where*

$$(2.8) \quad S = \{\alpha_j : n+1 \leq j \leq 2n\}$$

and

$$(2.9) \quad R = \{x_{p,q} : n+1 \leq p < q \leq 2n\}$$

where

$$x_{p,q} = \begin{cases} \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} & \text{if } c_{p-n,q-n} = 0 \\ \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} & \text{if } c_{p-n,q-n} = 1 \end{cases}.$$

Proof: Since $v_1 = e_1, \dots, v_n = e_n$ form a basis of $N \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and also pairwise form cones in Δ , we can apply [22, Lemma 3.2]. We can relate the above notations with those in the proof of [22, Lemma 3.2] as follows:

$$y_{j,\epsilon} = t_\epsilon \cdot \alpha_j \cdot t_\epsilon$$

for $n+1 \leq j \leq 2n$ and $\epsilon \in \mathbb{Z}_2^n$. Thus $y_{j,\epsilon} = \alpha_j^{\pm 1}$ by (2.4). Let $B^p := \epsilon + b^p$ and $B^q := \epsilon + b^q$ in \mathbb{Z}_2^n , where b^p and b^q are as defined in (2.3). Then it can be seen that

$$y_{p,\epsilon} \cdot y_{q,B^q} \cdot y_{p,B^p+B^q} \cdot y_{q,B^q} = t_\epsilon \cdot \alpha_p \cdot t_{b^p} \alpha_q t_{b^p} \cdot t_{b^q+b^p} \alpha_p t_{b^q+b^p} \cdot t_{b^q} \alpha_q t_{b^q} \cdot t_\epsilon$$

whenever $n+1 \leq p, q \leq 2n$. Moreover, by (2.4) (2.5), (2.6) and (2.7) we can further see that if $p < q$ and $c_{p-n,q-n} = 0$ then

$$\begin{aligned} \alpha_p \cdot t_{b^p} \alpha_q t_{b^p} \cdot t_{b^q+b^p} \alpha_p t_{b^q+b^p} \cdot t_{b^q} \alpha_q t_{b^q} &= (\alpha_q \cdot t_{b^q} \alpha_p t_{b^q} \cdot t_{b^q+b^p} \alpha_q t_{b^q+b^p} \cdot t_{b^p} \alpha_p t_{b^p})^{-1} \\ &= \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} \end{aligned}$$

and if $p < q$ and $c_{p-n,q-n} = 1$ then

$$\begin{aligned} \alpha_p \cdot t_{b^p} \alpha_q t_{b^p} \cdot t_{b^q+b^p} \alpha_p t_{b^q+b^p} \cdot t_{b^q} \alpha_q t_{b^q} &= (\alpha_q \cdot t_{b^q} \alpha_p t_{b^q} \cdot t_{b^q+b^p} \alpha_q t_{b^q+b^p} \cdot t_{b^p} \alpha_p t_{b^p})^{-1} \\ &= \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1}. \end{aligned}$$

Also it can be seen that when $p = q$ we simply get a trivial relation. Thus by [22, Lemma 3.2] it follows that that $\pi_1(Y_n)$ has a presentation with generators

$$(2.10) \quad S = \{\alpha_j : n+1 \leq j \leq 2n\}$$

and the relations

$$R' = \{x_{p,q}^\epsilon : n+1 \leq p < q \leq 2n\}$$

where

$$(2.11) \quad x_{p,q}^\epsilon = \begin{cases} t_\epsilon \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} t_\epsilon & \text{if } c_{p-n,q-n} = 0 \\ t_\epsilon \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} t_\epsilon & \text{if } c_{p-n,q-n} = 1 \end{cases}$$

for $\epsilon \in \mathbb{Z}_2^n$. Moreover, it can also be seen that each $x_{p,q}^\epsilon$ is conjugate to either $x_{p,q}$ or $x_{p,q}^{-1}$ by an element of the free group on S . For instance, in the case when $c_{p-n} = 0$ and $c_{q-n} = 1$ and $c_{p-n,q-n} = 1$ we have:

$$\begin{aligned} x_{p,q}^\epsilon &= t_\epsilon \cdot \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} \cdot t_\epsilon \\ &= t_\epsilon \alpha_p t_\epsilon \cdot t_\epsilon \alpha_q^{-1} t_\epsilon \cdot t_\epsilon \alpha_p^{-1} t_\epsilon \cdot t_\epsilon \alpha_q^{-1} t_\epsilon \\ &= \alpha_p \alpha_q \alpha_p^{-1} \alpha_q \\ &= (\alpha_q^{-1}) \cdot (\alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot (\alpha_q) \\ &= \alpha_q^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_q. \end{aligned}$$

The other cases can be proved similarly (see Appendix). Thus it follows that R given by (2.9) is a complete set of relations for $\pi_1(Y_n)$. Hence the theorem. \square

Corollary 2.2. *The group $\pi_1(Y_n)$ is abelian if and only if Y_n is a product of n -copies of \mathbb{P}^1 .*

Proof: From the above theorem we see that $\pi_1(Y_n)$ is abelian if and only if $c_{p-n, q-n} = 0$ for all $n+1 \leq p < q \leq 2n$. Hence the corollary. \square

Lemma 2.3. *Y_n is an aspherical manifold.*

Proof: Note that Y_n is an \mathbb{S}^1 -bundle over Y_{n-1} so by the long exact homotopy sequence we get:

$$(2.12) \quad \dots \rightarrow \pi_2(Y_{n-1}) \rightarrow \pi_1(\mathbb{S}^1) \rightarrow \pi_1(Y_n) \rightarrow \pi_1(Y_{n-1}) \rightarrow 0 \dots$$

Since $\pi_k(Y_1) = 0$ for $k \geq 2$, the proof follows by induction on n . \square

Corollary 2.4. *The group $\pi_1(Y_n)$ is always torsion free.*

Proof: This follows readily since Y_n is an aspherical manifold. \square

Remark 2.5. Lemma 2.3 alternately follows from [22, Theorem 1.2], since it can easily be seen that the fan Δ associated to Y_n is *flag like*. More precisely, we cannot find three edge vectors $\{v_i, v_j, v_k\}$ which pairwise form a cone in Δ but together do not form a cone in Δ . Also, Corollary 2.2 can alternately be derived from the combinatorial criterion on Δ for the fundamental group to be abelian in [22, Theorem 5.1].

2.1. Further group theoretic properties of $\pi_1(Y_n)$.

Proposition 2.6. *The commutator subgroup $[\pi_1(Y_n), \pi_1(Y_n)]$ is abelian. In particular, $\pi_1(Y_n)$ is a solvable group.*

Proof: For every v_i , $1 \leq i \leq n$ (resp. $n+1 \leq i \leq 2n$) there exists a unique v_{i+n} (resp. v_{i-n}) such that v_i, v_{i+n} (resp. v_i, v_{i-n}) do not form a cone in Δ . Thus by [22, Lemma 4.1], $[W, W]$ is abelian. Further, since $\pi_1(Y_n)$ is a subgroup of W (see 2.2) it follows that $[\pi_1(Y_n), \pi_1(Y_n)]$ is a subgroup of $[W, W]$ and is hence abelian. Now, $1 \leq [\pi_1, \pi_1] \leq \pi_1(Y_n)$ gives an abelian tower for $\pi_1(Y_n)$ so that $\pi_1(Y_n)$ is solvable. \square

Let $\overline{\alpha_j}$ denote the image of α_j under the canonical abelianization homomorphism

$$\pi_1(Y_n) \rightarrow H_1(Y_n; \mathbb{Z}) \simeq \pi_1(Y_n) / [\pi_1(Y_n), \pi_1(Y_n)].$$

We then have the following description of $H_1(Y_n; \mathbb{Z})$:

Proposition 2.7. *The group $H_1(Y_n; \mathbb{Z})$ has a presentation with generators*

$$(2.13) \quad \langle \overline{\alpha_j} : n+1 \leq j \leq 2n \rangle$$

and relations

$$(2.14) \quad \overline{\alpha_p} \cdot \overline{\alpha_q} \cdot \overline{\alpha_p}^{-1} \cdot \overline{\alpha_q}^{-1}$$

for $n+1 \leq p, q \leq 2n$ and

$$(2.15) \quad \overline{\alpha_q}^{-2}$$

for those $n+1 \leq q \leq 2n$ for which there exists a $p < q$ such that $c_{p-n, q-n} = 1$. Thus additively we have an isomorphism $H_1(Y_n; \mathbb{Z}) \simeq \mathbb{Z}^{n-r} \oplus \mathbb{Z}_2^r$ where r is the number of $n+1 \leq q \leq 2n$ for which there exists a $p < q$ with $c_{p-n, q-n} = 1$.

Corollary 2.8. *The commutator subgroup $[\pi_1(Y_n), \pi_1(Y_n)]$ is a free abelian group with generators α_q^2 where $n+1 \leq q \leq 2n$ is such that there exists $p < q$ with $c_{p-n, q-n} = 1$.*

Proof: In $[W, W]$, $w \cdot [s_i, s_j] \cdot w^{-1} = [s_j, s_i]$ if the reduced word w contains either s_i or s_j but not both, $w \cdot [s_i, s_j] \cdot w^{-1} = [s_i, s_j]$ otherwise. But we know that $[s_i, s_j] = (s_i \cdot s_j)^2 = 1$ when $j \neq i+n$. So, $[W, W] = \langle w \cdot [s_i, s_j] \cdot w^{-1} \mid 1 \leq i, j \leq d, w \in W \rangle = \langle (s_i \cdot s_{i+n})^2 \mid 1 \leq i \leq n \rangle$. Observe that $\alpha_j^2 = (s_j \cdot s_1^{c_{j-n,1}} \cdots s_n^{c_{j-n,n}})^2 = (s_j \cdot s_{j-n})^2$ for $j = n+1 \leq j \leq 2n$. Thus $[W, W]$ is generated by $\langle \alpha_j^2 \mid n+1 \leq j \leq 2n \rangle$ as a subgroup of $\pi_1(Y_n)$. Moreover, by [22, Lemma 4.2], the torsion elements of W and hence in $[W, W]$ are only of order 2. Hence it follows that $[W, W]$ is a free abelian group generated by $\alpha_j^2, n+1 \leq j \leq 2n$. Since $[\pi_1(Y_n), \pi_1(Y_n)]$ is a subgroup of $[W, W]$, by Proposition 2.7, the corollary follows. \square

Proposition 2.9. *The group $\pi_1(Y_n)$ is nilpotent if and only if it is abelian.*

Proof: Let $\overline{\alpha_q^2}$, $n+1 \leq q \leq 2n$ is such that there exists a $p < q$ such that $c_{p-n, q-n} = 1$. Then we see that $\alpha_q^2 \alpha_p \alpha_q^{-2} \alpha_p^{-1} = (s_q s_{q-n})^4 = \alpha_q^4$, which belongs to $[\pi_1(Y_n), [\pi_1(Y_n), \pi_1(Y_n)]]$. Proceeding similarly by induction we get that

$$\alpha_q^{2(k-1)} \alpha_p \alpha_q^{-2(k-1)} \alpha_p^{-1} = \alpha_q^{2k}$$

belongs to $\pi_1(Y_n)^{(k)}$. Here $\pi_1(Y_n)^{(1)} := [\pi_1(Y_n), \pi_1(Y_n)]$ and $\pi_1(Y_n)^{(k)} := [\pi_1(Y_n), \pi_1(Y_n)^{(k-1)}]$. Since α_q is of infinite order in W , the proposition follows. \square

Remark 2.10. In particular, by Corollary 2.2, $\pi_1(Y_n)$ is nilpotent if and only if Y_n is a product of n -copies of \mathbb{P}^1 's.

Remark 2.11. Here we mention that the fundamental group of a real toric variety is not in general solvable. For example, the non-orientable surfaces of genus g are real toric varieties (see for example [18, Section 4.5, Remark 4.5.2] and [22, Remark 3.3]), whose fundamental groups contain free subgroups of rank $g-1$ (see for example [2, p. 62, Section 4]). Thus whenever $g \geq 3$, these groups are not solvable. Hence the property proved in Proposition 2.6 is specific to the real Bott tower.

3. COHOMOLOGY RING WITH \mathbb{Z}_2 -COEFFICIENTS

In this section we shall describe cohomology ring $H^*(Y_n; \mathbb{Z}_2)$ in terms of generators and relations by applying [6, Theorem 5.12], viewing Y_n as a small cover. More precisely, we state the following theorem.

Theorem 3.1. *Let $\mathcal{R} := \mathbb{Z}_2[x_1, x_2, \dots, x_{2n}]$ and let \mathcal{I} denote the ideal in \mathcal{R} generated by the following set of elements*

$$(3.16) \quad \{x_i x_{n+i}, x_i - x_{n+i} + \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+j} \mid 1 \leq i \leq n\}.$$

As a graded \mathbb{Z}_2 -algebra $H^(Y_n; \mathbb{Z}_2)$ is isomorphic to \mathcal{R}/\mathcal{I} .*

Proof: The proof immediately follows by applying [6, Theorem 4.13] viewing Y_n as a small cover over an n -cube with the characteristic map given by means of the Bott matrix. Thus we get the following isomorphism of graded \mathbb{Z}_2 -algebras:

$$(3.17) \quad \psi : \mathcal{R}/\mathcal{I} \simeq H^*(Y_n; \mathbb{Z}_2)$$

where ψ maps $x_i + I$ to the fundamental class of the *characteristic submanifold* $[M_i] \in H^1(Y_n; \mathbb{Z}_2)$ for $1 \leq i \leq 2n$. Hence the theorem. \square

Corollary 3.2. *Let $\tilde{\mathcal{R}} := \mathbb{Z}_2[y_1, y_2, \dots, y_n]$ and let $\tilde{\mathcal{I}}$ denote the ideal in $\tilde{\mathcal{R}}$ generated by the following set of elements*

$$(3.18) \quad \{y_i^2 - \sum_{j=1}^{i-1} c_{j,i} \cdot y_i y_j \mid 1 \leq i \leq n\}.$$

We have the following isomorphism of graded \mathbb{Z}_2 -algebras:

$$(3.19) \quad \tilde{\psi} : \tilde{\mathcal{R}}/\tilde{\mathcal{I}} \simeq H^*(Y_n; \mathbb{Z}_2)$$

where $\tilde{\psi}$ maps $y_i + I$ to the fundamental class of the characteristic submanifold $[M_{i+n}] \in H^1(Y_n; \mathbb{Z}_2)$ for $1 \leq i \leq n$.

Proof: In (3.16), by substituting the second relation in the first, we can simplify the presentation of \mathcal{R}/\mathcal{I} by reducing the generators to x_{n+1}, \dots, x_{2n} and the relations to

$$(3.20) \quad x_{n+i} \cdot [x_{n+i} - \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+j}] \mid 1 \leq i \leq n.$$

Now, by sending y_i to x_{i+n} we get the required isomorphism from \mathcal{R}/\mathcal{I} to $\tilde{\mathcal{R}}/\tilde{\mathcal{I}}$

Remark 3.3. We can also apply results in [18, Section 4.3] to get Theorem 3.1 by viewing Y_n as a real toric variety.

Remark 3.4. Note that, in fact there exists a canonical line bundle $[L_i]$ on Y_n such that $[M_i] = w_1(L_i)$ for every $1 \leq i \leq 2n$, where $w_1(L_i)$ denotes the first Stiefel Whitney class of L_i . Thus the cohomology ring is generated by $w_1(L_i)$ for $1 \leq i \leq 2n$ (see [6, Section 6.1]).

4. STIEFEL WHITNEY CLASSES OF Y_n

Let $w_k(Y_n)$ denote the k th Stiefel-Whitney class of Y_n for $0 \leq k \leq n$ with the understanding that $w_0(Y_n) = 1$. Then $w(Y_n) = 1 + w_1(Y_n) + \dots + w_n(Y_n)$ is the total Stiefel-Whitney class of Y_n .

Theorem 4.1. (i) *Under the isomorphism (3.17) of $H^*(Y_n; \mathbb{Z}_2)$ with \mathcal{R}/\mathcal{I} we have the identification*

$$(4.21) \quad w(Y_n) = \prod_{i=1}^{2n} (1 + x_i)$$

where x_i for $1 \leq i \leq 2n$ satisfy (3.16).

(ii) *We further have the following recursive formula*

$$(4.22) \quad w(Y_n) = w(Y_{n-1}) \cdot (1 + x_n)(1 + x_{2n}),$$

where

$$(4.23) \quad x_n \cdot x_{2n} = 0, x_n = x_{2n} - \sum_{i=1}^{n-1} c_{i,n} x_{n+i}.$$

Proof: The proof of (i) follows readily by applying [6, Corollary 6.8] for Y_n .

Now, we shall prove (ii).

Note that the defining Bott matrix for Y_{n-1} is the $(n-1) \times (n-1)$ submatrix of C obtained by deleting the n th row and the n th column of C .

Moreover, let $\pi_n : Y_n \rightarrow Y_{n-1}$ denote the canonical projection of the \mathbb{RP}^1 -bundle. Then via pullback along π_n^* , $H^*(Y_{n-1}; \mathbb{Z}_2)$ can be identified with the subring $\mathcal{R}'/\mathcal{I}'$ of \mathcal{R}/\mathcal{I} where $\mathcal{R}' = \mathbb{Z}_2[x_1, x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_{2n-1}]$ and \mathcal{I}' is the ideal generated by the relations

$$(4.24) \quad \{x_i x_{n+i}, x_i - x_{n+i} + \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+j} \text{ for } 1 \leq i \leq n-1\}.$$

Since Y_n is an \mathbb{RP}^1 -bundle over Y_{n-1} , we further have the following presentation of $H^*(Y_n; \mathbb{Z}_2)$ as an algebra over $H^*(Y_{n-1}; \mathbb{Z}_2)$:

$$(4.25) \quad H^*(Y_n; \mathbb{Z}_2) \simeq H^*(Y_{n-1}; \mathbb{Z}_2)[x_n, x_{2n}]/J$$

where J is the ideal generated by the relations

$$(4.26) \quad x_n \cdot x_{2n}, x_n - x_{2n} + \sum_{i=1}^{n-1} c_{i,n} x_{n+i}.$$

Furthermore, via π_n^* we can identify $w(Y_{n-1})$ with the expression

$$(4.27) \quad w(Y_{n-1}) = \prod_{i=1}^{n-1} (1 + x_i) \cdot \prod_{i=n+1}^{2n-1} (1 + x_i)$$

in \tilde{R} where x_i for $1 \leq i \leq n-1$ and $n+1 \leq i \leq 2n-1$ satisfy the relations (4.24). Now by (4.21) and (4.27), (ii) follows. \square

Corollary 4.2.

(i) The following hold in the \mathbb{Z}_2 -algebra \mathcal{R}/\mathcal{I} :

$$(4.28) \quad w(Y_n) = w(Y_{n-1}) \left(1 + \sum_{i=1}^{n-1} c_{i,n} x_{n+i}\right),$$

$$(4.29) \quad w_k(Y_n) = w_k(Y_{n-1}) + w_{k-1}(Y_{n-1}) \cdot \left(\sum_{i=1}^{n-1} c_{i,n} x_{n+i}\right)$$

for $n \geq 2$ and $1 \leq k \leq n$.

(ii) For every $1 \leq k \leq n$, $w_k(Y_n)$ is a \mathbb{Z}_2 -linear combination of square free monomials of degree k in the variables x_{n+1}, \dots, x_{2n-1} modulo \mathcal{I} .

Proof: The equation (4.21) reduces to (4.28) by applying (4.23). Note that under the isomorphism ψ of graded algebras $H^*(Y_n; \mathbb{Z}_2)$ and \mathcal{R}/\mathcal{I} , $w_k(Y_n) \in H^k(Y_n; \mathbb{Z}_2)$ corresponds to a polynomial of degree k in $x_i, 1 \leq i \leq 2n$ modulo \mathcal{I} for $1 \leq k \leq n$. Thus we get (4.29) by comparing the degree k -terms on either side of (4.28) and (i) follows.

Observe that by applying (4.24), in $\mathcal{R}'/\mathcal{I}'$ and hence in \mathcal{R}/\mathcal{I} , we can substitute for x_i in terms of x_{n+1}, \dots, x_{n+i} modulo \mathcal{I} using the equality

$$(4.30) \quad x_i = x_{n+i} + \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+j}.$$

In particular, $w_k(Y_{n-1})$ (resp. $w_{k-1}(Y_{n-1})$) can be written as a polynomial of degree k (resp. $k-1$) in x_{n+i} , $1 \leq i \leq n-1$. Furthermore, multiplying either side of (4.30) with x_{n+i} , along with the equality $x_i \cdot x_{n+i} = 0$ gives

$$(4.31) \quad x_{n+i}^2 = \sum_{j=1}^{i-1} c_{j,i} \cdot x_{n+i} \cdot x_{n+j}$$

for $1 \leq i \leq n-1$. It follows that $w_k(Y_{n-1})$ (resp. $w_{k-1}(Y_{n-1})$) can be expressed as \mathbb{Z}_2 -linear combinations of square free monomials of degree k (resp. $k-1$) in x_{n+i} , $1 \leq i \leq n-1$ in the algebra \mathcal{R}/\mathcal{I} . Now, assertion (ii) follows readily by applying (4.31) again in (4.29). \square

Corollary 4.3. *We have the following elegant formula for $w_{n-1}(Y_n)$ in $H^*(Y_n; \mathbb{Z}_2)$ in terms of the Bott numbers $c_{i,j}$:*

$$w_{n-1}(Y_n) = c_{1,2} \cdot c_{2,3} \cdots c_{n-1,n} \cdot x_{n+1} \cdot x_{n+2} \cdots x_{2n-1}.$$

Proof: The proof follows by induction on n and (4.29) using the fact that in $H^*(Y_n; \mathbb{Z}_2)$ the following relations hold :

$$(4.32) \quad x_{n+1}^2 = 0; x_{n+1} \cdot x_{n+2}^2 = 0; x_{n+1} \cdot x_{n+2} \cdot x_{n+3}^2 = 0 \cdots; x_{n+1} \cdot x_{n+2} \cdots x_{2n-2}^2 = 0.$$

\square

4.1. Orientability of the real Bott tower. In this section we give a necessary and sufficient condition for Y_n to be orientable.

Lemma 4.4. *We have the following expression for the $w_1(Y_n)$ in \mathcal{R}/\mathcal{I} :*

$$(4.33) \quad w_1(Y_n) = \sum_{i=1}^{n-1} \left(\sum_{j=i+1}^n c_{i,j} \right) \cdot x_{n+i}.$$

Proof: The lemma follows by putting $k=1$ in (4.29) followed by induction on n . \square

Theorem 4.5. *The space Y_n is orientable if and only if*

$$(4.34) \quad \sum_{j=i+1}^n c_{i,j} \equiv 0 \pmod{\mathbb{Z}_2} \text{ for every } 1 \leq i \leq n-1,$$

where $c_{i,j}$ are the entries of the defining Bott matrix C (see 1.1).

Proof: Note that via the isomorphism (3.17) of graded algebras \mathcal{R}/\mathcal{I} and $H^*(Y_n; \mathbb{Z}_2)$ $\{x_{n+1}, x_{n+2}, \dots, x_{2n}\}$ corresponds to a basis over \mathbb{Z}_2 of $H^1(Y_n; \mathbb{Z}_2)$. Thus by (4.33), it follows that $w_1(Y_n) = 0$ if and only if

$$(4.35) \quad \sum_{j=i+1}^n c_{i,j} \equiv 0 \pmod{\mathbb{Z}_2} \text{ for every } 1 \leq i \leq n-1.$$

Furthermore, since a necessary and sufficient condition for a compact connected differentiable manifold M to be orientable is $w_1(M) = 0$, the theorem follows. \square

Corollary 4.6. *Let Y_n be an oriented real Bott tower Y_n . Then $w_{n-1}(Y_n) = 0$.*

Proof: By (4.34) it follows that $c_{n-1,n} = 0$ if Y_n is orientable. Now by (4.3) the corollary follows. \square

Remark 4.7. The assertion of Corollary 4.6 is true for any even dimensional manifold but not in general true when the dimension is odd (see [14, Theorem II and examples on p. 94]). Our assertion although specific to the case of a real Bott tower, holds in all dimensions.

Remark 4.8. In particular, a 3-step oriented real Bott tower Y_3 satisfies $w_2(Y_3) = 0$, and hence admits spin structure. This is a special case of the well known more general result of Steenrod that an oriented threefold is parallelizable.

Remark 4.9. Trivially product of n copies of \mathbb{P}^1 's is a parallelizable Bott tower for any n . Converse is not true as can be seen by the 3-step Bott tower associated with Bott numbers $c_{1,2} = 1$, $c_{1,3} = 1$ and $c_{2,3} = 0$, which is parallelizable but not a product of \mathbb{P}^1 's.

Now we give a combinatorial characterization for Y_n to admit a spin structure.

Theorem 4.10. *The orientable Bott tower Y_n admits a spin structure if and only if in addition to (4.34) the following identities hold for $1 \leq j < k \leq n-2$:*

$$(4.36) \quad \sum_{r=j+1}^n \sum_{\substack{s=k+1 \\ s \neq r}}^n c_{j,r} \cdot c_{k,s} + c_{j,k} \sum_{\substack{r,s=k+1 \\ r < s}}^n c_{k,r} \cdot c_{k,s} \equiv 0 \pmod{\mathbb{Z}_2}$$

where $c_{i,j}$ are as defined in (1.1).

Proof: By Theorem 3.1 and by equation (4.21), $w(Y_n)$ can be identified with the class in \mathcal{R}/\mathcal{I} of the following term

$$(4.37) \quad \prod_{j=1}^n (1 + x_j + x_{n+j} + x_j \cdot x_{n+j}).$$

Further, using the relations (3.16) in \mathcal{I} we can rewrite (4.37) as

$$(4.38) \quad \prod_{j=2}^n \left(1 + \sum_{i=1}^{j-1} c_{i,j} \cdot x_{n+i} \right)$$

Furthermore, since Theorem 3.1 gives an isomorphism of graded \mathbb{Z}_2 -algebras, the degree 2 term of $w(Y_n)$, namely $w_2(Y_n)$ can be identified with the degree 2 term of expression (4.38) which is the class of the following term in \mathcal{R}/\mathcal{I} :

$$(4.39) \quad \sum_{1 \leq j < k \leq n-1} \left(\sum_{r=j+1}^n \sum_{\substack{s=k+1 \\ s \neq r}}^n c_{j,r} c_{k,s} \right) x_{n+j} x_{n+k} + \sum_{k=1}^{n-2} \left(\sum_{\substack{r,s=k+1 \\ r < s}}^n c_{k,r} c_{k,s} \right) x_{n+k}^2.$$

By substituting $x_{n+k}^2 = \sum_{j=1}^{k-1} c_{j,k} \cdot x_{n+j} \cdot x_{n+k}$ from (4.31) and $c_{n-1,n} = 0$ from (4.34) in (4.39), we get that $w_2(Y_n)$ can be identified with the class of the following term in \mathcal{R}/\mathcal{I}

$$(4.40) \quad \sum_{1 \leq j < k \leq n-2} \left(\sum_{r=j+1}^n \sum_{\substack{s=k+1 \\ s \neq r}}^n c_{j,r} c_{k,s} + c_{j,k} \cdot \sum_{\substack{r,s=k+1 \\ r < s}}^n c_{k,r} c_{k,s} \right) x_{n+j} x_{n+k}.$$

Further, as a graded \mathbb{Z}_2 -vector space $H^2(Y_n; \mathbb{Z}_2)$ is isomorphic to the subspace of \mathcal{R}/\mathcal{I} freely generated over \mathbb{Z}_2 by the classes of $x_{n+j} x_{n+k}$, $1 \leq j < k \leq n$. Moreover,

the necessary and sufficient condition for an orientable manifold M to admit a spin structure is $w_2(M) = 0$. Thus the theorem follows from (4.40). \square

Example 4.11. The 4-step Bott towers admitting spin structure are classified by the following list of associated Bott matrices.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark 4.12. Note that the above list of Bott matrices exhausts all orientable 4-step Bott towers. Thus it follows that every orientable 4-step Bott tower is also spin. Moreover, it is known that a 4-manifold M is parallelizable if and only if it admits a spin structure (i.e. $w_1(M) = w_2(M) = 0$) and has vanishing Euler characteristic and signature ($\chi(M) = \sigma(M) = 0$) (see [11, Section 4] and [19, p. 699]). Moreover, by Hirzebruch signature formula, $\sigma(M) = \frac{1}{3}p_1(M)[M]$, where $p_1(M)$ is the first Pontrjagin class and $[M]$ the fundamental class of M . Now, a real Bott tower has vanishing Euler characteristic (see Remark 4.22) and vanishing Pontrjagin classes by ([6, Corollary 6.8 (i)]). Thus it follows that a 4-step Bott tower is orientable if and only if it is parallelizable. Further, it corresponds to one of the eight Bott matrices in the above list.

The following example shows that this is not the case in dimensions 5 and higher. Indeed there are n -step Bott towers which are orientable but not spin when $n \geq 5$.

Example 4.13. Let Y_n be the n -step Bott tower, $n \geq 5$, associated to the Bott numbers $c_{1,2} = 1$, $c_{1,n-2} = 1$; $c_{n-2,n-1} = 1$, $c_{n-2,n} = 1$; $c_{i,j} = 0$ for $i \neq 1, n-2$ and $c_{1,j} = 0$ for $j \neq 2, n-2$. These numbers clearly satisfy (4.34) but not (4.36). Indeed in this case, when $j = 1$ and $k = n-2$, the left hand side of (4.36) is $c_{1,2}(c_{n-2,n-1} + c_{n-2,n}) + c_{1,n-2}(c_{n-2,n-1} + c_{n-2,n}) + c_{1,n-2}c_{n-2,n-1}c_{n-2,n} \equiv 1 \pmod{\mathbb{Z}_2}$.

Definition 4.14. We call the Bott matrix C spin if and only if the associated Bott tower $Y_n = Y(C)$ is spin.

Let R_i denote the i th row vector $(0, \dots, 0, 1 = c_{i,i}, c_{i,i+1}, c_{i,i+2}, \dots, c_{i,n})$ of C . For every $1 \leq j < k \leq n$, we define an $n \times n$ Bott matrix C_{jk} with R_j as the j th row and R_k as the k th row. Further, for $i \neq j, k$, we let the i th row of C_{jk} have 1 as the (i, i) th entry and all other entries as 0.

Corollary 4.15. The Bott matrix C is spin if and only if C_{jk} is spin for every $1 \leq j < k \leq n-2$.

Proof: From Theorem 4.10 a necessary and sufficient condition for C to be spin is that the entries $c_{i,j}$, $i+1 \leq j \leq n$ on the row R_i for every $1 \leq i \leq n$ satisfy (4.34) and further, the entries $c_{j,r}$, $j+1 \leq r \leq n$ of R_j and $c_{k,s}$, $k+1 \leq s \leq n$ of R_k for every $1 \leq j < k \leq n$ satisfy (4.36).

Again by Theorem 4.10 it follows that, the necessary and sufficient condition for the Bott matrix C_{jk} to be spin is that the entries $c_{j,r}$, $j+1 \leq r \leq n$ of the j th row and the entries $c_{k,s}$, $k+1 \leq s \leq n$ of the k th row of C_{jk} , satisfy (4.34) and (4.36). This can be readily seen because any row of C_{jk} other than the j th or k th row has all entries as 0 except the diagonal entry which is 1. Thus the entries above the main diagonal on the i th row of C_{jk} where $i \neq j, k$, trivially satisfy (4.34). Moreover, if either $i \neq j, k$ or $l \neq j, k$ and $1 \leq i < l \leq n$, the entries of C_{jk} above the main diagonal, on the i th and the l th row trivially satisfy (4.36). Hence the corollary. \square

Remark 4.16. See [12] for results on criterion for Y_n to admit spin structure using different methods. We however note here that the main result [12, Theorem 1.2] follows immediately from Corollary 4.15 above. This can be seen because the conditions (4.34) and (4.36) need to be checked only for the entries of pairs of nonzero rows of the matrix A where $A := C - I$. In [12], A is called the Bott matrix. Thus we need to check the spin condition only on those $A_{jk} := C_{jk} - I$ with j th and k th nonzero rows.

Moreover, our result is more general as we do not require the assumption in [12, Theorem 1.2] that the number of nonzero rows of the matrix A is even (see [12, Remark 2.1]).

We illustrate Corollary 4.15 by the following examples.

Example 4.17.

(1)

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{23}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{24}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{34}}$$

Clearly $\sum_{j=i+1}^n c_{i,j} \equiv 0 \pmod{\mathbb{Z}_2}$ for all $1 \leq i \leq 5$.

When $j = 2$ and $k = 3$ the left hand side of (4.10) is

$$c_{2,3}(c_{3,4} + c_{3,5} + c_{3,6}) + c_{2,4}(c_{3,5} + c_{3,6}) + c_{2,5}(c_{3,4} + c_{3,6}) + c_{2,6}(c_{3,4} + c_{3,5}) + c_{2,3}(c_{3,4}c_{3,5} + c_{3,4}c_{3,6} + c_{3,5}c_{3,6}) \equiv 0 \pmod{\mathbb{Z}_2}$$

When $j = 3$ and $k = 4$ the left hand side of (4.10) is

$$c_{3,4}(c_{4,5} + c_{4,6}) + c_{3,5}c_{4,6} + c_{3,6}c_{4,5} + c_{3,4} \cdot c_{4,5} \cdot c_{4,6} \equiv 0 \pmod{\mathbb{Z}_2}$$

When $j = 2$ and $k = 4$ the left hand side of (4.10) is

$$c_{2,3}(c_{4,5} + c_{4,6}) + c_{2,4}(c_{4,5} + c_{4,6}) + c_{2,5}c_{4,6} + c_{2,6}c_{4,5} + c_{2,4}c_{4,5}c_{4,6} \equiv 0 \pmod{\mathbb{Z}_2}.$$

Since C_{23}, C_{24}, C_{34} are all spin by Corollary 4.15, C is spin. We do not consider the matrices C_{1l} for $2 \leq l \leq 4$, since the first row of C has nonzero entry only on the main diagonal.

(2)

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{12}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{13}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{23}}$$

Clearly $\sum_{j=i+1}^n c_{i,j} = 0$ for all $1 \leq i \leq 4$.

When $j = 1$ and $k = 2$ the left hand side of (4.10) is

$$c_{1,2}(c_{2,3} + c_{2,4} + c_{2,5}) + c_{1,3}(c_{2,4} + c_{2,5}) + c_{1,4}(c_{2,3} + c_{2,5}) + c_{1,5}(c_{2,3} + c_{2,4}) + c_{1,2}(c_{2,3}c_{2,4} + c_{2,3}c_{2,5} + c_{2,4}c_{2,5}) \equiv 0 \pmod{\mathbb{Z}_2}.$$

When $j = 1$ and $k = 3$ the left hand side of (4.10) is

$$c_{1,2}(c_{3,4} + c_{3,5}) + c_{1,3}(c_{3,4} + c_{2,5}) + c_{1,4}c_{3,5} + c_{1,5}c_{3,4} + c_{1,3}c_{3,4}c_{3,5} \equiv 1 \pmod{\mathbb{Z}_2}.$$

Thus C is not spin since C_{13} is not spin.

(3)

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{23}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{24}} \quad \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{C_{34}}$$

Clearly $\sum_{j=i+1}^n c_{i,j} = 0$ for all $1 \leq i \leq 6$.

When $j = 2$ and $k = 3$ the left hand side of (4.10) is

$$c_{2,3}(c_{3,4} + c_{3,5} + c_{3,6} + c_{3,7}) + c_{2,4}(c_{3,5} + c_{3,6} + c_{3,7}) + c_{2,5}(c_{3,4} + c_{3,6} + c_{3,7}) + c_{2,6}(c_{3,4} + c_{3,5} + c_{3,7}) +$$

$$c_{2,3}(c_{3,4}c_{3,5} + c_{3,4}c_{3,6} + c_{3,4}c_{3,7} + c_{3,5}c_{3,6} + c_{3,5}c_{3,7} + c_{3,6}c_{3,7}) \equiv 1 \pmod{\mathbb{Z}_2}.$$

Thus C is not spin since C_{23} is not spin.

4.2. Existence of a nowhere vanishing tangent vector field.

Proposition 4.18. *The n -step Bott tower is the total space of a fibre bundle over \mathbb{S}^1 with fibre an $n-1$ -step Bott tower corresponding to the Bott matrix C^1 of size $n-1 \times n-1$, defined by deleting the first row and first column of C .*

Proof: This can be seen as follows: Let N' denote the lattice $\bigoplus_{i=2}^n \mathbb{Z}e_i$. Let $v'_1 = e_2, v'_2 = e_3, \dots, v'_{n-1} = e_n$. Also let $v'_{n+1} = -e_2 + c_{2,3}e_3 + \dots + c_{2,n}e_n, \dots, v'_{2n-2} = -e_{n-1} + c_{n-1,n}e_n$ and $v'_{2n-1} = -e_n$.

We define the fan Δ' in N' consisting of cones generated by the set of vectors in any subcollection of $\{v'_1, v'_2, \dots, v'_{n-1}, v'_{n+1}, \dots, v'_{2n-1}\}$ which does not contain both v'_i and v'_{n+i} for $1 \leq i \leq n-1$.

Also let N'' denote the lattice $\mathbb{Z}e_1$ and Δ'' denote the fan consisting of the one dimensional cones generated by the vectors e_1 and $-e_1$ and the cone $\{0\}$ which corresponds to the real toric variety $\mathbb{P}_{\mathbb{R}}^1$.

By projecting to $\mathbb{Z}e_1$ we get an exact sequence of fans

$$(4.41) \quad 0 \rightarrow (\Delta', N') \rightarrow (\Delta, N) \rightarrow (\Delta'', N'') \rightarrow 0.$$

Moreover, the fan $\widetilde{\Delta}''$ in N consisting of the one dimensional cones generated by the vectors e_1 and $-e_1 + c_{1,2}e_2 + \dots + c_{1,n}e_n$ and the cone $\{0\}$ is a lift of Δ'' . Moreover, it can be seen that every cone σ of (N, Δ) is a sum $\sigma' + \sigma''$ of a cone in (N', Δ') and a cone in $(N, \widetilde{\Delta}'')$.

Thus we see that $Y_n = X(\Delta, N)$ is a toric fibre bundle over \mathbb{S}^1 with fibre an $n-1$ -step real Bott tower corresponding to the fan (N', Δ') (see [8, p. 41]). \square

In this convention, we shall denote the n -step Bott tower by Z_n and the fibre, which is the $n-1$ -step real Bott tower associated to the matrix C^1 and hence the fan (N', Δ') , by Z_{n-1} . We can iterate this process and view Z_{n-1} again as a fibre bundle over \mathbb{S}^1 with fibre Z_{n-2} which is the $n-2$ -step real Bott tower associated to the matrix C^2 of size $n-2 \times n-2$, obtained by deleting the first and the second rows and columns of C . Continuing this process $n-2$ times we finally get that Z_2 is a two step Bott tower associated to the Bott matrix C^{n-2} , obtained by deleting the first $n-2$ rows and columns of C . Then Z_2 is a fibre bundle over \mathbb{S}^1 with fibre $Z_1 \simeq \mathbb{S}^1$.

Following the above convention, in this section we shall denote the n -step real Bott tower by Z_n . Let $p_n : Z_n \rightarrow \mathbb{S}^1$ denote the projection of this fibre bundle.

Theorem 4.19. *The n -step real Bott tower Z_n has a nowhere vanishing continuous tangent vector field. Moreover, the tangent bundle splits into a direct sum of line bundles.*

Proof: Note that the tangent bundle of Z_n is the direct sum of the pull back of the tangent bundle of \mathbb{S}^1 and the relative tangent bundle of Z_{n-1} .

Since the tangent bundle of \mathbb{S}^1 is trivial its pull back to Z_n is a trivial line bundle. Thus a nowhere vanishing section of the trivial line bundle gives a nowhere vanishing section for the tangent bundle of Z_n .

Furthermore, if we assume by induction that the tangent bundle of an $n-1$ -step Bott tower is a direct sum of line bundles, then the relative tangent bundle of Z_{n-1} in the above fibration is also a direct sum of the associated line bundles. It follows that the tangent bundle of the n -step Bott tower is a direct sum of n -line bundles. Hence the proof. \square

Corollary 4.20. *Let Z_n be an orientable real Bott tower. Then the Euler class of Z_n vanishes.*

Proof: The proof is an immediate consequence of the [17, Theorem 4.19 and p. 101]. \square

Corollary 4.21. *The n -step real Bott tower Z_n is orientable (respectively spin) implies that the successive fibres $Z_{n-1}, Z_{n-2}, \dots, Z_2$ in the above iterated construction are all orientable (respectively spin).*

Proof: This follows from (4.34) and (4.36) since the Bott matrix corresponding to Z_k is C^{n-k} which is the matrix obtained from C by deleting the first k rows and k columns. \square

Remark 4.22. Alternately the assertion of Corollary 4.20 follows more directly from the fact that the Euler characteristic is multiplicative for the total space of fibre bundles. Indeed $\chi(Z_n) = 0$ since $\chi(\mathbb{S}^1) = 0$. In fact this also implies the statement on existence of nowhere vanishing vector field in Theorem 4.19 by Hopf's theorem.

4.3. Real Bott-towers bound.

Definition 4.23. *A smooth compact n -dimensional manifold M without boundary is null cobordant if it is diffeomorphic to the boundary of some compact smooth $n + 1$ -dimensional manifold \mathcal{W} with boundary.*

Let $w_k := w_k(Y_n)$ for $1 \leq k \leq n$. Also let μ_{Y_n} denote the fundamental class of Y_n in $H_n(Y_n; \mathbb{Z}_2)$. Then

$$(4.42) \quad \langle w_1^{r_1} \cdots w_n^{r_n}, \mu_{Y_n} \rangle \in \mathbb{Z}_2$$

such that $\sum_{i=1}^n i \cdot r_i = n$ are the Steifel-Whitney numbers of Y_n .

Theorem 4.24. *The n -step Bott tower Y_n is null-cobordant.*

Proof: From Corollary 4.2 (ii), it follows that, any monomial $w_1^{r_1} \cdots w_n^{r_n}$ of total dimension n , under the isomorphism ψ corresponds in \mathcal{R}/\mathcal{I} , to a \mathbb{Z}_2 -linear combination of square free monomials of degree n in $x_{n+1}, x_{n+2}, \dots, x_{2n-1}$. But there are no square free monomials of degree n in x_{n+j} , $1 \leq j \leq n-1$. Thus the monomial $w_1^{r_1} \cdots w_n^{r_n} = 0$ in $H^n(Y_n; \mathbb{Z}_2)$ so that the associated Stiefel-Whitney number is zero. Therefore by Thom's theorem it follows that Y_n is null-cobordant. \square

Definition 4.25. *A smooth compact n -dimensional manifold M' without boundary is orientedly null cobordant if it is diffeomorphic to the boundary of some compact smooth $n + 1$ -dimensional oriented manifold \mathcal{W}' with boundary.*

Let Y_n denote an oriented n -step real Bott tower. Let $p_i := p_i(Y_n)$ denote the i th Pontrjagin class of Y_n in $H^{4i}(Y_n, \mathbb{Z})$ and μ_{Y_n} denote the fundamental homology class in $H_n(Y_n, \mathbb{Z})$. Then for each $I = i_1, \dots, i_r$ a partition of k , the I th Pontrjagin number of Y_n is given by

$$(4.43) \quad \langle p_{i_1} \cdots p_{i_r}, \mu_{Y_n} \rangle \in \mathbb{Z}$$

when $n = 4k$. It is zero when n is not divisible by 4 (see [17, p. 185]).

Corollary 4.26. *Let Y_n be an oriented n -step real Bott tower, then it is orientedly null-cobordant.*

Proof: Note that [6, Corollary 6.8 (i)], implies that all the Pontrjagin numbers of Y_n vanish. Moreover, we have shown above in the proof of Theorem 4.24 that all the Stiefel-Whitney numbers of Y_n vanish. Thus the corollary follows by Wall's theorem ([23, Section 8, Corollary 1]). \square

Remark 4.27. There are examples of real toric varieties whose top Stiefel-Whitney class does not vanish. For example the non-orientable surfaces of odd genus are real toric varieties (see [18, Section 4.5, Remark 4.5.2] [22, Remark 3.3]) having non-vanishing second Stiefel-Whitney class. Thus the properties we have proved in this and the preceding section are all specific to real Bott towers.

Remark 4.28. Another motivation for this work was to relate the topology of the real Bott tower with the topology of the real Bott Samelson manifolds using the degeneration results of Grossberg and Karshon (see [13]). This could further be used to obtain results on the topology of real flag manifolds. In particular, to study the fundamental group and cohomology ring of the real Bott Samelson manifolds and real flag variety. This shall again be taken up in future work.

Remark 4.29. During the process of this work the authors also came across the work of Kamishima and Masuda [20] where among other results they also compute the fundamental group and cohomology ring of real Bott towers but using different methods.

APPENDIX

Proof that $x_{p,q}^\epsilon$ is a conjugate of either $x_{p,q}$ or $x_{p,q}^{-1}$ by an element of the free group generated by S :

- (1) $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 0$ and $\epsilon_{q-n} = 0$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1} \\ &= x_{p,q} \end{aligned}$$

- (2) $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 0$ and $\epsilon_{q-n} = 1$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q \\ &= \alpha_q^{-1} \cdot (\alpha_q \alpha_p \alpha_q^{-1} \alpha_p^{-1}) \cdot \alpha_q \\ &= \alpha_q^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot \alpha_q \\ &= \alpha_q^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_q \end{aligned}$$

- (3) $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 1$ and $\epsilon_{q-n} = 0$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p^{-1} \alpha_q \alpha_p \alpha_q^{-1} \\ &= \alpha_p^{-1} \cdot (\alpha_q \alpha_p \alpha_q^{-1} \alpha_p^{-1}) \cdot \alpha_p \\ &= \alpha_p^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot \alpha_p \\ &= \alpha_p^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_p \end{aligned}$$

- (4) $c_{p-n,q-n} = 0$, $\epsilon_{p-n} = 1$ and $\epsilon_{q-n} = 1$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p^{-1} \alpha_q^{-1} \alpha_p \alpha_q \\ &= (\alpha_p \alpha_q)^{-1} \cdot (\alpha_p \alpha_q \alpha_p^{-1} \alpha_q^{-1}) \cdot (\alpha_p \alpha_q) \\ &= (\alpha_p \alpha_q)^{-1} \cdot x_{p,q} \cdot (\alpha_p \alpha_q) \end{aligned}$$

- (5) $c_{p-n,q-n} = 1$, $\epsilon_{p-n} = 0$ and $\epsilon_{q-n} = 0$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1} \\ &= x_{p,q} \end{aligned}$$

- (6) $c_{p-n,q-n} = 1$, $\epsilon_{p-n} = 1$ and $\epsilon_{q-n} = 0$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p^{-1} \alpha_q^{-1} \alpha_p \alpha_q^{-1} \\ &= (\alpha_p \alpha_q^{-1})^{-1} \cdot (\alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1}) \cdot (\alpha_p \alpha_q^{-1}) \\ &= (\alpha_p \alpha_q^{-1})^{-1} \cdot x_{p,q} \cdot (\alpha_p \alpha_q^{-1}) \end{aligned}$$

- (7) $c_{p-n,q-n} = 1$, $\epsilon_{p-n} = 1$ and $\epsilon_{q-n} = 1$:

$$\begin{aligned} x_{p,q}^\epsilon &= \alpha_p^{-1} \alpha_q \alpha_p \alpha_q \\ &= \alpha_p^{-1} \cdot (\alpha_q \alpha_p \alpha_q \alpha_p^{-1}) \cdot \alpha_p \\ &= \alpha_p^{-1} \cdot (\alpha_p \alpha_q^{-1} \alpha_p^{-1} \alpha_q^{-1})^{-1} \cdot \alpha_p \\ &= \alpha_p^{-1} \cdot x_{p,q}^{-1} \cdot \alpha_p \end{aligned}$$

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